ME 261: Numerical Analysis

Lecture-9&10: Numerical Interpolation

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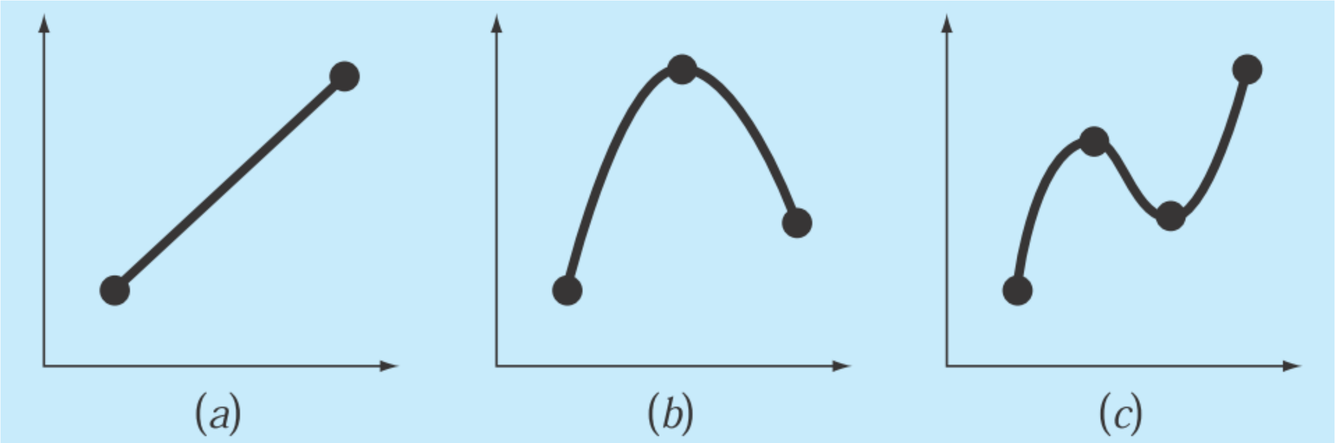
- To estimate **intermediate values** between **precise data points** most commonly used method used is **polynomial interpolation**
- For n+1 data points, there is **only one polynomial of order n** that passes through **all n+1 data points**

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots\dots + a_n x^n$$

- There are two different ways regarding the formats of the polynomial expression, namely:
  - **The Newton polynomial of interpolation**
  - **The Langrange polynomial of interpolation**



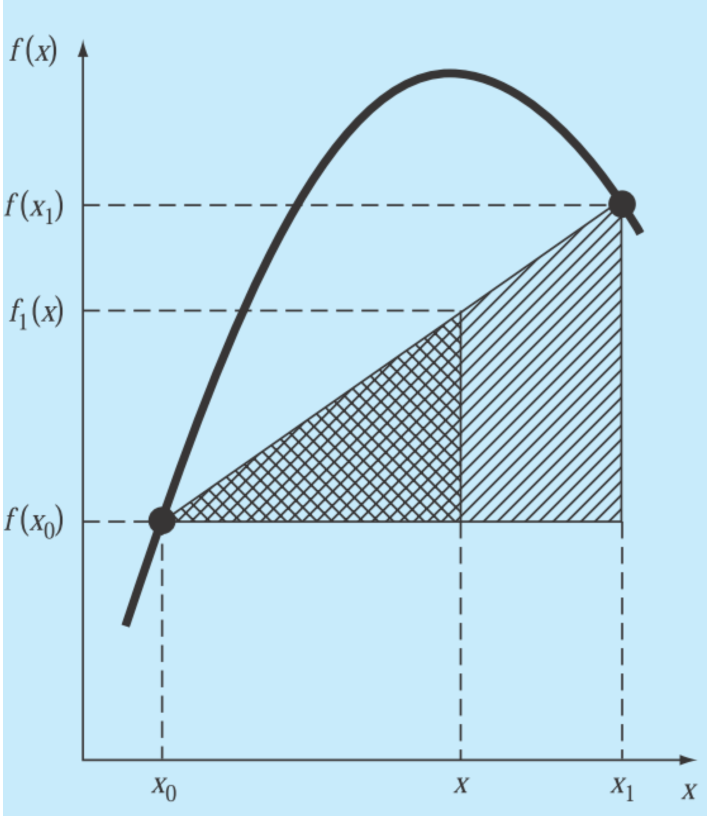
(a) first-order (linear) connecting two points, (b) second order (quadratic or parabolic) connecting three points(c) third-order (cubic) connecting four points.

Newton's Divided-Difference Interpolating polynomial

- **Linear Interpolation**
- The **simplest** form of interpolation
- The interpolating polynomial is of **first order/linear** (i.e. two points are necessary)

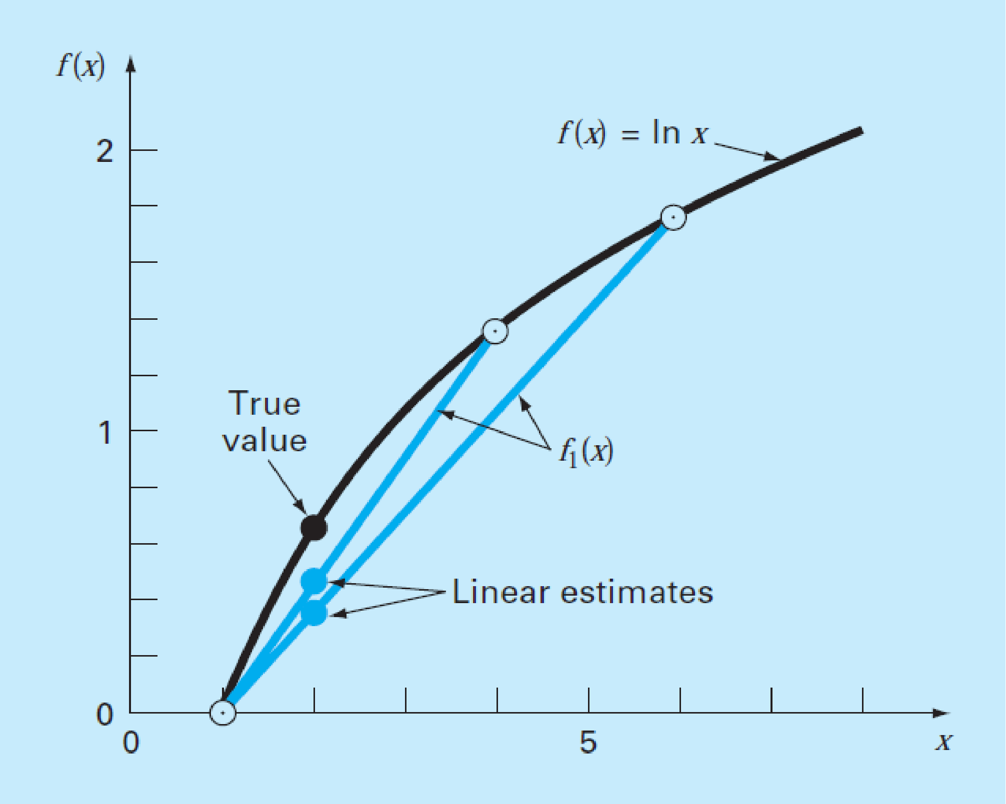
$$\frac{f_1(x)-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$f_1(x)=f(x_0)+\frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$$



- The **slope** of the interpolating polynomial is the **finite divided-difference approximation** of the first derivative
- The **smaller** the interval, the **better** the approximation

- Estimate the natural logarithm of 2 using linear interpolation.



Two linear interpolations to estimate ln 2. Note how the smaller interval provides a better estimate.

Newton's Divided-Difference Interpolating polynomial

- **Quadratic Interpolation**
- Linear approximation is **very raw**
- The **accuracy** of interpolation can be **improved** by introducing **higher order** interpolation polynomial **if more data points** are known
- Let us assume **three data points are known at x_0, x_1 and x_2**

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$f_2(x) = b_0 + b_1 x - b_1 x_0 + b_2\left(x^2 - x x_1 - x x_0 + x_0 x_1\right)$$

$$= b_0 + b_1 x - b_1 x_0 + b_2 x^2 - b_2 x x_1 - b_2 x x_0 + b_2 x_0 x_1$$

$$= b_0 - b_1 x_0 + b_2 x_0 x_1 + (b_1 - b_2 x_0 - b_2 x_1)x + b_2 x^2$$

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Our goal

$$a_0 = b_0 - b_1 x_0 + b_2 x_0 x_1$$

$$a_1 = b_1 - b_2 x_0 - b_2 x_1$$

$$a_2 = b_2$$

To determine the constants b_0, b_1, and b_2

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$at\ x = x_0; \quad f(x_0) = b_0;$$

$$at\ x = x_1; \quad f(x_1) = b_0 + b_1(x_1 - x_0);$$

$$at\ x = x_2; \quad f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1);$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\begin{aligned}
f(x_2) &= b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\
&= b_0 + b_1(x_2 - x_1 + x_1 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\
&= b_0 + b_1(x_2 - x_1) + b_1(x_1 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\
&= b_0 + b_1(x_2 - x_1) + f(x_1) - f(x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\
&= f(x_0) + b_1(x_2 - x_1) + f(x_1) - f(x_0) + b_2(x_2 - x_0)(x_2 - x_1) \\
&= b_1(x_2 - x_1) + f(x_1) + b_2(x_2 - x_0)(x_2 - x_1)
\end{aligned}$$

$$f(x_2) - f(x_1) = b_1(x_2 - x_1) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} = b_1 + b_2(x_2 - x_0)$$

$$b_2(x_2 - x_0) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - b_1 = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)}$$

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{x_2 - x_0}$$

$$f_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_1) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{x_2 - x_0}(x - x_0)(x - x_1)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

- **Second order interpolation equation contains one extra term in comparison to that of a first order interpolation equation that introduces second order curvature**

Fit a second-order polynomial to the three points. Use the polynomial to evaluate ln 2.

$x_0 = 1 \qquad f(x_0) = 0$

$x_1 = 4 \qquad f(x_1) = 1.386294$

$x_2 = 6 \qquad f(x_2) = 1.791759$

General Form of Newton's Interpolating polynomial

- **Nth order Interpolation polynomial**
- To fit an nth-order polynomial for n + 1 data points as:

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

- Known Data points **$[x_0, f(x_0)], [x_1, f(x_1)], \ldots, [x_n, f(x_n)]$** can be used to evaluate the coefficients **$b_0, b_1, \ldots, b_n$** as:

$$b_0 = f[x_0] = f(x_0)$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_n = f[x_n, x_{n-1}, \cdots, x_1, x_0]$$

- **The bracketed function evaluations $f[x_i, x_j]$ are finite divided differences**

- Example, the first finite divided difference is represented generally as

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

- The *second finite divided difference, which represents the difference of two first divided* differences, is expressed generally as

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

- Similarly, the *nth finite divided difference is*

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$\ldots$$

$$b_n = f[x_n, x_{n-1}, \ldots\ldots x_{1,} x_0] = \frac{f[x_n, x_{n-1,} \ldots\ldots x_1] - f[x_{n-1}, x_{n-2,} \ldots\ldots x_0]}{x_n - x_0}$$

b_1: First Finite divided difference **[$\approx f'(x)$]**

b_2: Second Finite divided difference **[$\approx f''(x)$]**

b_n: n^{th} Finite divided difference

i	x_i	$f(x_i)$	First	Second	Third
0	x_0	$f(x_0)$	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$	$f[x_3, x_2, x_1, x_0]$
1	x_1	$f(x_1)$	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	
2	x_2	$f(x_2)$	$f[x_3, x_2]$		
3	x_3	$f(x_3)$			

FIGURE 18.5
Graphical depiction of the recursive nature of finite divided differences.

$$f[x_j, x_{j-1}, \ldots, x_{i+1}, x_i] = \frac{f[x_j, \ldots, x_{i+1}] - f[x_{j-1}, \ldots, x_i]}{x_j - x_i}$$

Example

Construct a 4th order polynomial in Newton form that passes through the following points:

i	0	1	2	3	4
x_i	0	1	-1	2	-2
$f(x_i)$	-5	-3	-15	39	-9

4th order polynomial:

$$f_4(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$f_4(x) = b_0 + b_1(x - 0) + b_2(x - 0)(x - 1) + b_3(x - 0)(x - 1)(x + 1) + b_4(x - 0)(x - 1)(x + 1)(x - 2)$$

$$f_4(x) = b_0 + b_1(x) + b_2(x)(x - 1) + b_3(x)(x - 1)(x + 1) + b_4(x)(x - 1)(x + 1)(x - 2)$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	$f[x_i]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	0	-5				
1	1	-3				
2	-1	-15				
3	2	39				
4	-2	-9				

$$f[x_i] = f(x_i)$$

i	0	1	2	3	4
x_i	0	1	-1	2	-2
$f(x_i)$	-5	-3	-15	39	-9

			Divided Difference (b_n)			
			First	Second	Third	Fourth
i	x_i	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	0	-5	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$	$f[x_3, x_2, x_1, x_0]$	$f[x_4, x_3, x_2, x_1, x_0]$
1	1	-3	$f[x_2, x_1]$	$f[x_3, x_2, x_1]$	$f[x_4, x_3, x_2, x_1]$	
2	-1	-15	$f[x_3, x_2]$	$f[x_4, x_3, x_2]$		
3	2	39	$f[x_4, x_3]$			
4	-2	-9				

$$f[x_j, x_{j-1}, \ldots, x_{i+1}, x_i] = \frac{f[x_j, \ldots, x_{i+1}] - f[x_{j-1}, \ldots, x_i]}{x_j - x_i}$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	0	-5	2			
1	1	-3	6			
2	-1	-15	18			
3	2	39	12			
4	-2	-9				

$$f[x_1,x_0]=\frac{f[x_1]-f[x_0]}{x_1-x_0}=\frac{-3-(-5)}{1-0}=2$$

$$f[x_2,x_1]=\frac{f[x_2]-f[x_1]}{x_2-x_1}=\frac{-15-(-3)}{-1-1}=6$$

$$f[x_3,x_2]=\frac{f[x_3]-f[x_2]}{x_3-x_2}=\frac{39-(-15)}{2-(-1)}=18$$

$$f[x_4,x_3]=\frac{f[x_4]-f[x_3]}{x_4-x_3}=\frac{-9-(39)}{-2-2}=12$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	f[ ]	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
0	0	-5	2	-4		
1	1	-3	6	12		
2	-1	-15	18	6		
3	2	39	12			
4	-2	-9				

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{6-2}{-1-0} = -4$$

$$f[x_3, x_2, x_1] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1} = \frac{18-6}{2-1} = 12$$

$$f[x_4, x_3, x_2] = \frac{f[x_4, x_3] - f[x_3, x_2]}{x_4 - x_2} = \frac{12-18}{-2-(-1)} = 6$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	0	-5	2	-4	8	
1	1	-3	6	12	2	
2	-1	-15	18	6		
3	2	39	12			
4	-2	-9				

$$f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = \frac{12-(-4)}{2-0} = 8$$

$$f[x_4, x_3, x_2, x_1] = \frac{f[x_4, x_3, x_2] - f[x_3, x_2, x_1]}{x_4 - x_1} = \frac{6-12}{-2-1} = 2$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	f[ ]	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
0	0	-5	2	-4	8	3
1	1	-3	6	12	2	
2	-1	-15	18	6		
3	2	39	12			
4	-2	-9				

$$f[x_4, x_3, x_2, x_1, x_0]$$

$$= \frac{f[x_4, x_3, x_2, x_1] - f[x_3, x_2, x_1, x_0]}{x_4 - x_0}$$

$$= \frac{2-8}{-2-0} = 3$$

To calculate b_0, b_1, b_2, b_3, we can construct a divided difference table as

i	x_i	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	0	-5	2	-4	8	3
1	1	-3 b_0	6 b_1	12 b_2	2 b_3	b_4
2	-1	-15	18	6		
3	2	39	12			
4	-2	-9				

Thus the polynomial is:

$$f_4(x) = -5 + 2(x) - 4(x)(x-1) + 8(x)(x-1)(x+1)$$
$$+3(x)(x-1)(x+1)(x-2)$$

Errors of Newton's Interpolating Polynomials

Notice that the structure of Eq. is similar to the Taylor series expansion in the sense that terms are added sequentially to capture the higher-order behavior of the underlying function.

Consequently, as with the Taylor series, if the true underlying function is an *nth-order polynomial, the nth-order interpolating polynomial based on n + 1* data points will yield exact results.

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$$

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where *ξ is somewhere in the interval containing the unknown and the data.*

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\cdots(x - x_n)$$